Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 55 (2008) 485–493
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Improvement and further generalization of inequalities for differentiable mappings and applications

Uğur S. Kirmaci

Atatürk University, K.K. Education Faculty, Department of Mathematics, 25240 Erzurum, Turkey

Received 27 November 2006; received in revised form 26 April 2007; accepted 16 May 2007

Abstract

Based on the celebrated Hermite–Hadamard integral inequality for convex functions, some inequalities for differentiable convex and concave mappings are generalized. Furthermore, the results obtained are examined in the context of special means for real numbers.
Published by Elsevier Ltd

Keywords: Hermite–Hadamard inequality; Convex functions; Concave functions; Integral inequalities; Differentiable mappings

1. Introduction

Let $f : I \subset R \to R$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as the Hermite–Hadamard inequality for convex functions [1].

In [2,3], the author established the following results connected with the left-hand side of (1.1) and applied them to obtain some elementary inequalities for real numbers and in numerical integration.

Theorem A. *Let $f : I^o \subset R \to R$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$, and let $p > 1$. If the mapping $\left|f'\right|^{p/(p-1)}$ is convex on $[a, b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{1.2}$$

E-mail address: kirmaci@atauni.edu.tr.

0898-1221/$ - see front matter Published by Elsevier Ltd
doi:10.1016/j.camwa.2007.05.004

Theorem B. *Let $f : I^o \subset R \to R$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$, and let $p > 1$. If the mapping $|f'|^p$ is convex on $[a, b]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right)\right| \le \left(\frac{3^{1-(1/p)}}{8}\right)(b-a)\left(|f'(a)| + |f'(b)|\right). \tag{1.3}$$

Theorem C. *Let $f : I^o \subset R \to R$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$, and let $p > 1$. If $|f'|^p$ is concave on $[a, b]$ and $|f'|$ is a linear map, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right)\right| \le \left(\frac{b-a}{8}\right)\left(|f'(a+b)|\right). \tag{1.4}$$

Dragomir and Agarwal [4] established the following result connected with the right-hand side of (1.1).

Theorem D. *Let $f : I^o \subset R \to R$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$, and let $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{2(p+1)^{1/p}}\left[\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2}\right]^{(p-1)/p}. \tag{1.5}$$

Pearce and Pecaric [5] established the following results:

Theorem E. *Let $f : I^o \subset R \to R$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$, $f' \in L(a, b)$, and let $q \ge 1$. If the mapping $|f'|^q$ is convex on $[a, b]$, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{1/q}. \tag{1.6}$$

Theorem F. *Let $f : I^o \subset R \to R$ be a differentiable mapping on I^o, $a, b \in I^o$ with $a < b$, $f' \in L(a, b)$ and let $q \ge 1$. If $|f'|^q$ is concave on $[a, b]$, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{1.7}$$

The aim of this paper is to establish some improvements of Theorems A–F. In Section 3, we note some consequent applications to special means.

For several recent results concerning the Hermite–Hadamard inequality, we refer the reader to [6,7].

Throughout we suppose I is an interval on R and $a, b, c, A, B \in I^0$ with $a \le A \le c \le B \le b$. $(c \ne a, b)$, $p, q \in R$ and $f : I^0 \to R$ is differentiable.

2. Main results

Theorem 1. *Let $f : I^0 \subset R \to R$ be a differentiable mapping on I^0, $a, b \in I^0$, with $a < b$ and let $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then we have*

i.

$$\begin{aligned} &f(ca+(1-c)b)(B-A) + f(a)(1-B) + f(b)A - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\ &\quad = (a-b)\left[\int_0^c (t-A)f'(ta+(1-t)b)\mathrm{d}t + \int_c^1 (t-B)f'(ta+(1-t)b)\mathrm{d}t\right]. \end{aligned} \tag{2.1}$$

ii.

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+(b)A-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]\right|$$
$$\leq\left[\frac{A^{p+1}+(c-A)^{p+1}}{p+1}\right]^{1/p}\left(\frac{c^2\left|f'(a)\right|^q+(2c-c^2)\left|f'(b)\right|^q}{2}\right)^{1/q}$$
$$+\left[\frac{(B-c)^{p+1}+(1-B)^{p+1}}{p+1}\right]^{1/p}\left(\frac{(1-c^2)\left|f'(a)\right|^q+(1-c)^2\left|f'(b)\right|^q}{2}\right)^{1/q}. \tag{2.2}$$

Proof. i. Let $S:[a,b]\to R$ be defined by

$$S(t)=\begin{cases}t-A, & t\in[0,c]\\ t-B, & t\in(c,1].\end{cases}$$

Integrating by parts and using the change of the variable $x=ta+(1-t)b$, we have

$$\int_0^1 S(t)f'(ta+(1-t)b)\mathrm{d}t=\int_0^c(t-A)f'(ta+(1-t)b)\mathrm{d}t+\int_c^1(t-B)f'(ta+(1-t)b)\mathrm{d}t$$
$$=\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right].$$

ii. From equality (2.1), we deduce

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]\right|$$
$$\leq\int_0^c|t-A|\left|f'(ta+(1-t)b)\right|\mathrm{d}t+\int_c^1|t-B|\left|f'(ta+(1-t)b)\right|\mathrm{d}t. \tag{2.3}$$

For $p>1$, it follows from Hölder's inequality that

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]\right|$$
$$\leq\left(\int_0^c|t-A|^p\,\mathrm{d}t\right)^{1/p}\left(\int_0^c\left|f'(ta+(1-t)b)\right|^q\mathrm{d}t\right)^{1/q}$$
$$+\left(\int_c^1|t-B|^p\,\mathrm{d}t\right)^{1/p}\left(\int_c^1\left|f'(ta+(1-t)b)\right|^q\mathrm{d}t\right)^{1/q} \tag{2.4}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Using the convexity of $\left|f'\right|^q$, we obtain

$$\int_0^c\left|f'(ta+(1-t)b)\right|^q\mathrm{d}t\leq\int_0^c\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]\mathrm{d}t=\frac{c^2\left|f'(a)\right|^q+(2c-c^2)\left|f'(b)\right|^q}{2} \tag{2.5}$$

and

$$\int_c^1\left|f'(ta+(1-t)b)\right|^q\mathrm{d}t\leq\int_c^1\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]\mathrm{d}t$$
$$=\frac{(1-c^2)\left|f'(a)\right|^q+(1-c)^2\left|f'(b)\right|^q}{2}. \tag{2.6}$$

Also, we have

$$\int_c^1 t\mathrm{d}t = \frac{1-c^2}{2}, \qquad \int_c^1 (1-t)\mathrm{d}t = \frac{(1-c)^2}{2}, \qquad \int_0^c (1-t)\mathrm{d}t = \frac{2c-c^2}{2}, \qquad \int_0^c t\mathrm{d}t = \frac{c^2}{2},$$
$$\int_0^c |t-A|^p\,\mathrm{d}t = \int_0^A (A-t)^p\mathrm{d}t + \int_A^c (t-A)^p\mathrm{d}t = \frac{A^{p+1}+(c-A)^{p+1}}{p+1}, \tag{2.7}$$
$$\int_c^1 |t-B|^p\,\mathrm{d}t = \int_c^B (B-t)^p\mathrm{d}t + \int_B^1 (t-B)^p\mathrm{d}t = \frac{(B-c)^{p+1}+(1-B)^{p+1}}{p+1}.$$

A combination of (2.4)–(2.7) gives the required inequality (2.2). □

Corollary 1. *Under the assumptions of Theorem 1 with $A = B = c = \frac{1}{2}$, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\leq (b-a)\left(\frac{1}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}}\right)\left[\left[\frac{|f'(a)|^q + 3|f'(b)|^q}{8}\right]^{1/q} + \left[\frac{3|f'(a)|^q + |f'(b)|^q}{8}\right]^{1/q}\right].$$

Let $a_1 = |f'(a)|^q$, $b_1 = 3|f'(b)|^q$, $a_2 = 3|f'(a)|^q$, $b_2 = |f'(b)|^q$. Here $0 \leq (p-1)/p < 1$, for $p > 1$. Using the fact that

$$\sum_{k=1}^n (a_k+b_k)^s \leq \sum_{k=1}^n a_k^s + \sum_{k=1}^n b_k^s, \tag{2.8}$$

for $(0 \leq s < 1)$, we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq (b-a)\left(\frac{1}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}}\right)\left[\frac{|f'(a)|+|f'(b)|}{2}\right]$$
$$\leq \left(\frac{b-a}{4}\right)\left(\frac{4}{p+1}\right)^{1/p}\left(|f'(a)|+|f'(b)|\right). \tag{2.9}$$

Corollary 2. *Under the assumptions of Theorem 1 with $A = 0$, $B = 1$, $c = \frac{1}{2}$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq (b-a)\left(\frac{1}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}}\right)\left[\left(\frac{|f'(a)|^q + 3|f'(b)|^q}{8}\right)^{1/q} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{8}\right)^{1/q}\right].$$

For $a_1 = |f'(a)|^q$, $b_1 = 3|f'(b)|^q$, $a_2 = 3|f'(a)|^q$, $b_2 = |f'(b)|^q$, using the inequality (2.8), we obtain inequality (1.2).

Remark 1. Inequality (2.9) is a refinement of inequality (1.5).

Theorem 2. *Let $f : I^0 \subset R \to R$ be a differentiable mapping on I^0, $a, b \in I^0$, with $a < b$ and let $p > 1$. If the mapping $|f'|^p$ is convex on $[a, b]$, then we have*

$$\left|f(ca+(1-c)b)(B-A) + f(a)(1-B) + f(b)A - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\leq (a-b)\left\{K^{1-\frac{1}{p}}\left[T|f'(a)|^p + (K-T)|f'(b)|^p\right]^{1/p} + M^{1-\frac{1}{p}}\left[N|f'(a)|^p + (M-N)|f'(b)|^p\right]^{1/p}\right\} \tag{2.10}$$

where

$$K = \frac{A^2 + (c-A)^2}{2}, \qquad T = \frac{A^3 + c^3}{3} - \frac{Ac^2}{2}, \qquad M = \frac{(B-c)^2 + (1-B)^2}{2},$$
$$N = \frac{B^3 + c^3 + 1}{3} - (1 + c^2)\frac{B}{2}.$$

Proof. Let us consider the inequality (2.3), i.e.,

$$\left|\frac{1}{a-b}\left[f(ca + (1-c)b)(B-A) + f(a)(1-B) + f(b)A - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]\right|$$
$$\leq \int_0^c |t-A| \left|f'(ta + (1-t)b)\right| \mathrm{d}t + \int_c^1 |t-B| \left|f'(ta + (1-t)b)\right| \mathrm{d}t.$$

By the power-mean inequality, we obtain

$$\int_0^c |t-A| \left|f'(ta + (1-t)b)\right| \mathrm{d}t \leq \left(\int_0^c |t-A| \,\mathrm{d}t\right)^{1-\frac{1}{p}} \left(\int_0^c |t-A| \left|f'(ta + (1-t)b)\right|^p \mathrm{d}t\right)^{1/p} \tag{2.11}$$

and

$$\int_c^1 |t-B| \left|f'(ta + (1-t)b)\right| \mathrm{d}t \leq \left(\int_c^1 |t-B| \,\mathrm{d}t\right)^{1-\frac{1}{p}} \left(\int_c^1 |t-B| \left|f'(ta + (1-t)b)\right|^p \mathrm{d}t\right)^{1/p}. \tag{2.12}$$

Since $\left|f'\right|^p$ is convex, we have

$$\begin{aligned}\int_0^c |t-A| \left|f'(ta + (1-t)b)\right|^p \mathrm{d}t &\leq \int_0^c |t-A| \left(t\left|f'(a)\right|^p + (1-t)\left|f'(b)\right|^p\right) \mathrm{d}t \\ &\leq T\left|f'(a)\right|^p + (K-T)\left|f'(b)\right|^p\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\int_c^1 |t-B| \left|f'(ta + (1-t)b)\right|^p \mathrm{d}t &\leq \int_c^1 |t-B| \left(t\left|f'(a)\right|^p + (1-t)\left|f'(b)\right|^p\right) \mathrm{d}t \\ &\leq N\left|f'(a)\right|^p + (M-N)\left|f'(b)\right|^p\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}K &= \int_0^c |t-A| \,\mathrm{d}t = \int_0^A (A-t)\mathrm{d}t + \int_A^c (t-A)\mathrm{d}t = \frac{A^2 + (c-A)^2}{2}, \\ T &= \int_0^c |t-A| \,t\mathrm{d}t = \int_0^A (A-t)t\mathrm{d}t + \int_A^c (t-A)t\mathrm{d}t = \frac{A^3 + c^3}{3} - \frac{Ac^2}{2} \\ M &= \int_c^1 |t-B| \,\mathrm{d}t = \int_c^B (B-t)\mathrm{d}t + \int_B^1 (t-B)\mathrm{d}t = \frac{(B-c)^2 + (1-B)^2}{2} \\ N &= \int_c^1 |t-B| \,t\mathrm{d}t = \int_c^B (B-t)t\mathrm{d}t + \int_B^1 (t-B)t\mathrm{d}t = \frac{B^3 + c^3 + 1}{3} - (1 + c^2)\frac{B}{2}\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}K - T &= \int_0^c |t-A| (1-t)\mathrm{d}t = \int_0^A (A-t)(1-t)\mathrm{d}t + \int_A^c (t-A)(1-t)\mathrm{d}t \\ M - N &= \int_c^1 |t-B| (1-t)\mathrm{d}t = \int_c^B (B-t)(1-t)\mathrm{d}t + \int_B^1 (t-B)(1-t)\mathrm{d}t.\end{aligned}$$

A combination of (2.3) and (2.11)–(2.15) gives the required inequality (2.10). □

Corollary 3. *Under the assumptions of Theorem 2 with $A = 0$, $B = 1$, $c = \frac{1}{2}$, we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq (b-a)\left(\frac{1}{8}\right)^{1-\frac{1}{p}}\left[\left(\frac{\left|f'(a)\right|^p + 2\left|f'(b)\right|^p}{24}\right)^{1/p} + \left(\frac{2\left|f'(a)\right|^p + \left|f'(b)\right|^p}{24}\right)^{1/p}\right].$$

For $a_1 = \left|f'(a)\right|^p, b_1 = 2\left|f'(b)\right|^p, a_2 = 2\left|f'(a)\right|^p, b_2 = \left|f'(b)\right|^p$, using the inequality (2.8), we obtain inequality (1.3).

Corollary 4. *Under the assumptions of Theorem 2 with $A = B = c = \frac{1}{2}$, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\leq \left(\frac{b-a}{8^{1-\frac{1}{p}}}\right)\left[\left[\frac{\left|f'(a)\right|^p + 5\left|f'(b)\right|^p}{48}\right]^{1/p} + \left[\frac{5\left|f'(a)\right|^p + \left|f'(b)\right|^p}{48}\right]^{1/p}\right].$$

For $a_1 = \left|f'(a)\right|^p, b_1 = 5\left|f'(b)\right|^p, a_2 = 5\left|f'(a)\right|^p, b_2 = \left|f'(b)\right|^p$, using the inequality (2.8), we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \leq \left(\frac{b-a}{8}\right)\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{2.16}$$

Remark 2. Inequality (2.16) is a refinement of inequality (1.6).

Theorem 3. *Let $f : I^0 \subset R \to R$ be a differentiable mapping on I^0, $a, b \in I^0$, with $a < b$ and let $p > 1$. If the mapping $\left|f'\right|^p$ is concave on $[a, b]$, then we have*

$$\left|f(ca+(1-c)b)(B-A) + f(a)(1-B) + f(b)A - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\leq (b-a)\left[K\left|f'\left(\frac{aT+b(K-T)}{K}\right)\right| + M\left|f'\left(\frac{aN+b(M-N)}{M}\right)\right|\right] \tag{2.17}$$

where

$$K = \frac{A^2+(c-A)^2}{2}, \qquad T = \frac{A^3+c^3}{3} - \frac{Ac^2}{2}, \qquad M = \frac{(B-c)^2+(1-B)^2}{2},$$
$$N = \frac{B^3+c^3+1}{3} - (1+c^2)\frac{B}{2}.$$

Proof. Let us consider the inequality (2.3), i.e.,

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A) + f(a)(1-B) + f(b)A - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]\right|$$
$$\leq \int_0^c |t-A|\left|f'(ta+(1-t)b)\right|\mathrm{d}t + \int_c^1 |t-B|\left|f'(ta+(1-t)b)\right|\mathrm{d}t.$$

Using the concavity of $\left|f'\right|^p$ and by the Jensen integral inequality, we have

$$\int_0^c |t-A|\left|f'(ta+(1-t)b)\mathrm{d}t\right| \leq \left(\int_0^c |t-A|\,\mathrm{d}t\right)\left|f'\left(\frac{\int_0^c |t-A|\,(ta+(1-t)b)\mathrm{d}t}{\int_0^c |t-A|\,\mathrm{d}t}\right)\right|$$
$$= K\left|f'\left(\frac{aT+b(K-T)}{K}\right)\right| \tag{2.18}$$

$$\int_c^1 |t-B|\left|f'(ta+(1-t)b)\mathrm{d}t\right| \le \left(\int_c^1 |t-B|\,\mathrm{d}t\right)\left|f'\left(\frac{\int_c^1 |t-B|\,(ta+(1-t)b)\mathrm{d}t}{\int_c^1 |t-B|\,\mathrm{d}t}\right)\right|$$
$$= M\left|f'\left(\frac{aN+b(M-N)}{M}\right)\right|. \tag{2.19}$$

A combination of (2.3), (2.15), (2.18) and (2.19) gives the required inequality (2.17). □

Corollary 5. *Under the assumptions of Theorem 3 with $A=0$, $B=1$, $c=\frac{1}{2}$ and if $\left|f'\right|$ is a linear map, we have inequality (1.4), i.e.,*

$$\left|\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right)\right| \le \left(\frac{b-a}{8}\right)\left(\left|f'(a+b)\right|\right).$$

Corollary 6. *Under the assumptions of Theorem 3 with $A=B=c=\frac{1}{2}$, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{(b-a)}{8}\left(\left|f'\left(\frac{a+5b}{6}\right)\right| + \left|f'\left(\frac{5a+b}{6}\right)\right|\right). \tag{2.20}$$

Remark 3. Since $\left|f'\right|^q$ being concave implies that $\left|f'\right|$ is concave, we have,

$$\frac{1}{2}\left(\left|f'\left(\frac{5a+b}{6}\right)\right| + \left|f'\left(\frac{a+5b}{6}\right)\right|\right) \le \left|f'\left(\frac{a+b}{2}\right)\right|.$$

Thus, inequality (2.20) is a refinement of inequality (1.7).

3. Applications to special means

As in [2], we shall consider the means for arbitrary real numbers $\alpha, \beta, \alpha \ne \beta$. We take

$$H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}, \quad \alpha,\beta \in R\setminus\{0\}, \text{ (harmonic mean)}$$
$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in R, \text{ (arithmetic mean)}$$
$$L(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta|-\ln|\alpha|}, \quad |\alpha| \ne |\beta|,\ \alpha\beta \ne 0, \text{ (logarithmic mean)}$$
$$I(\alpha,\beta) = \frac{1}{\mathrm{e}}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{1/(\beta-\alpha)}, \quad \text{(identric mean)}$$
$$G(\alpha,\beta) = \sqrt{\alpha\beta}, \quad \text{(geometric mean)}$$
$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{1/n}, \quad n \in Z\setminus\{-1,0\},\ \alpha,\beta \in R, \alpha \ne \beta, \text{ (generalized log–mean)}.$$

Now, using the results of Section 2, we give some applications to special means of real numbers.

Proposition 3.1. *Let $a, b \in I^0$, $0 < a < b$. Then the following holds, for all $p > 1$:*

$$|\ln A(a,b) - \ln I(a,b)| \le \frac{1}{4}\left(\frac{4}{p+1}\right)^{1/p}(b^2-a^2)G^{-2}(a,b)$$

and

$$\left|\ln I(a,b) - \frac{1}{2}\ln G^2(a,b)\right| \le \frac{1}{4}\left(\frac{4}{p+1}\right)^{1/p}(b^2-a^2)G^{-2}(a,b).$$

Proof. The assertions follow from Corollaries 1 and 2 applied to the convex mapping $f(x) = -\ln x$. $\square$

Proposition 3.2. *Let $a, b \in I^0$, $a < b$, $0 \notin [a, b]$ and $n \in Z$, $|n| \geq 2$. Then, for all $p > 1$,*

$$\left|L_{n-2}^{n-2}(a,b)G^{-2(2n-1)}(a,b) - A^{-2n}(a,b)\right| = \left|L_{-2n}^{-2n}(a,b) - A^{-2n}(a,b)\right| \leq 4|n|\left(\frac{3^{1-\frac{1}{p}}}{8}\right)(b-a)A\left(|a|^{-2n-1}, |b|^{-2n-1}\right)$$

and

$$\left|A(a^{-2n}, b^{-2n}) - L_{-2n}^{-2n}(a,b)\right| \leq n\left(\frac{b-a}{2}\right)A(|a|^{-2n-1}, |b|^{-2n-1}).$$

Proof. The assertions follow from Corollaries 3 and 4 applied to the convex mapping $f(x) = x^{-2n}, x \in R$, $n \in Z, |n| \geq 2$. $\square$

Proposition 3.3. *Let $a, b \in I^0$, $a < b$, $0 \notin [a, b]$ and $0 < n < 1$. Then, the following holds:*

$$\left|L_n^n(a,b) - A^n(a,b)\right| \leq |n|\left(\frac{b-a}{8}\right)\left(|a+b|^{n-1}\right)$$

and

$$\left|A(a^n, b^n) - L_n^n(a,b)\right| \leq n\left(\frac{b-a}{8}\right)\left|\left(\frac{a+5b}{6}\right)^{n-1} + \left(\frac{5a+b}{6}\right)^{n-1}\right|.$$

Proof. The assertions follow from Corollaries 5 and 6 applied to the concave mapping $f(x) = x^n$, $x \in [a, b]$, $0 < n < 1$. $\square$

Proposition 3.4. *Let $a, b \in I^0$, $0 < a < b$. Then the following holds, for all $p > 1$:*

$$|\ln A(a,b) - \ln I(a,b)| \leq \left(\frac{3^{1-\frac{1}{p}}}{8}\right)(b^2 - a^2)G^{-2}(a,b) \leq 2\left(\frac{3^{1-\frac{1}{p}}}{8}\right)(b-a)\,H^{-1}(a,b)$$

and

$$\left|\ln I(a,b) - \frac{1}{2}\ln G^2(a,b)\right| \leq \left(\frac{b^2-a^2}{8}\right)G^{-2}(a,b).$$

Proof. The assertions follow from Corollaries 3 and 4 applied to the convex mapping $f(x) = -\ln x$. $\square$

Proposition 3.5. *Let $a, b \in I^0$, $0 < a < b$. Then the following holds:*

$$|\ln A(a,b) - \ln I(a,b)| \leq \left(\frac{b-a}{16}\right)A^{-1}(a,b)$$

and

$$\left|\frac{1}{2}\ln G^2(a,b) - \ln I(a,b)\right| \leq \left(\frac{b-a}{4}\right)H^{-1}\left(\frac{a+5b}{6}, \frac{5a+b}{6}\right).$$

Proof. The assertions follow from Corollaries 5 and 6 applied to the concave mapping $f(x) = \ln x$. $\square$

References

[1] J.E. Pečarić, F. Proschan, Y.L. Tong, Convex Functions, Partial Ordering and Statistical Applications, Academic Press, New York, 1991.

[2] U.S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comput. 147 (2004) 137–146.

[3] U.S. Kırmacı, M.E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comput. 153 (2004) 361–368.

[4] S.S. Dragomir, R.P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett. 11 (5) (1998) 91–95.

[5] C.E.M. Pearce, J. Pecaric, Inequalities for differentiable mappings with application to special means and quadrature formula, Appl. Math. Lett. 13 (2000) 51–55.

[6] U.S. Kırmacı, M.E. Özdemir, Some inequalities for mappings whose derivatives are bounded and applications to special means of real numbers, Appl. Math. Lett. 17 (2004) 641–645.

[7] G.S. Yang, D.Y. Hwang, K.L. Tseng, Some inequalities for differentiable convex and concave mappings, Comput. Math. Appl. 47 (2004) 207–216.